The Method of Moments: A Numerical Technique for Wire Antenna Design

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This tutorial introduces the mathmatical foundations of the Method of Moments, a powerful tool for solving electromagnetic field problems such as antenna radiation and impedance

The Method of Moments technique, as applied to problems in electromagnetic theory, was introduced by Roger F. Harrington in his 1967 seminal paper, “Matrix Methods for Field Problems” [1]. The implementation of the Method of Moments, by Poggio and Burke at Lawrence Livermore National Labs during the 1970s, established this solution technique as a mainstay in the design of wire and wire array antennas.

This tutorial reviews the Method of Moments (MoM) from a practicing RF engineer’s perspective, with a view to providing understanding of its foundations, as opposed to rigorous mathematical exposition. The discussion begins with the formulation of Pocklington’s integral equation, an integral equation commonly used for wire antenna problems. The solution to Pocklington’s equation, using the MoM, is then explained. The integral equation solution yields the current distribution on the wire which, in turn, is used to calculate the antenna’s radiation characteristics and feed point impedance.

Introduction

Throughout the history of physical science, natural behaviors have been represented in terms of integro-differential equations. In many instances, behaviors are described in terms of simple differential equations.

$$\frac{dx}{dt} = v \tag{1}$$

where the function $x(t)$ is defined over the domain of t. The differential operator then yields the function $v(t)$ which is also defined over the domain of t.

In other instances, where the function $v(t)$ is known over the domain of t, specific values of x may be derived from representative expressions, such as Equation 2.

$$x = \int_0^{t_1} v(t)\,dt \tag{2}$$

For example, if $v(t) = k$, $x = kt_1$.

A special case arises when the function $v(t)$ is unknown and values of x are known at only discrete values of t. This type of problem is generally referred to as an integral equation problem where the task is to determine the function $v(t)$ with boundary conditions described by values of x at specific values of t.

The task of determining the current distribution on a wire antenna resulting from an arbitrary excitation may be readily stated in terms of an integral equation problem. The formulation begins with the development of an integral expression which defines the electric field resulting from an arbitrary current distribution on the wire. This integral expression will employ a Green’s function which relates the electric field at an arbitrary observation point to the current at an arbitrary source point. The integral equation problem then employs the integral expression to relate known electric field boundary conditions to an unknown current distribution on the wire.

The MoM applies orthogonal expansions to translate the integral equation statement into a system of circuit-like simultaneous linear equations. Basis functions are used to expand the current distribution. Testing functions are used to invoke the electric field boundary conditions. Matrix methods are then used to solve for the expansion coefficients associated with the basis functions. The current distribution solution is then constructed from the expansion coefficients. The antenna's radiation characteristics and feed point impedance are then derived from the calculated current distribution.

Pocklington's Integral Equation

A well-known formulation for simple wire antennas is Pocklington's integral equation. Figure 1 depicts a representative geometry from which Pocklington's equation can be derived. A simple wire antenna is positioned along the z axis in a Cartesian coordinate system. The current is restricted to the centerline of the wire and directed along the z axis. Elemental current segments are located at coordinate z'. Field observation points are located at coordinates z. A feedgap is positioned at $z = 0$. The electric field along the surface of the wire and in the feedgap, which establishes the boundary conditions for the problem, is defined as follows: $E_z = 0$ on the surface of the wire, $E_z = V_g/\Delta z$ at the feedgap. V_g, the antenna excitation, is normally set to 1.0 volts for input impedance calculations. Δz is commonly set equal the diameter of the wire. However, it is possible to study the impact of feedgap dimensions on antenna input impedance by varying the value of Δz. With the conditions presented in Figure 1, Pocklington's equation may be written as Equation 3.

$$\int_{-l/2}^{l/2} I(z')\left[\frac{\partial^2}{\partial z^2}+k^2\right]\frac{e^{-jkR}}{4\pi R}dz' = j\omega\varepsilon E_z(z) \quad (3)$$

where

$$R=\sqrt{\rho^2+(z-z')^2} \quad (4)$$

The variable R represents the distance between the current source and field observation points. The variable ρ specifies the radius of the wire. The current distribution $I_z(z')$ is defined along the length of the wire from $z' = l/2$ to $z' = -l/2$. The kernel $[\partial^2/\partial_z{}^2 + k^2]$ denotes the wave equation differential operator on the free space Green's function $e^{-jkR}/4\pi R$. The constant k specifies the free space wave number. $E_z(z)$ represents the electric field generated by the current on the wire.

With a specific excitation applied, as modeled through the appropriate boundary conditions, radiation characteristics and feedpoint impedances are determined from knowledge of the antenna's current distribution $I_z(z')$. Of the many techniques available to solve such integral equation problems, the Method of Moments is one of the industry's more popular approaches.

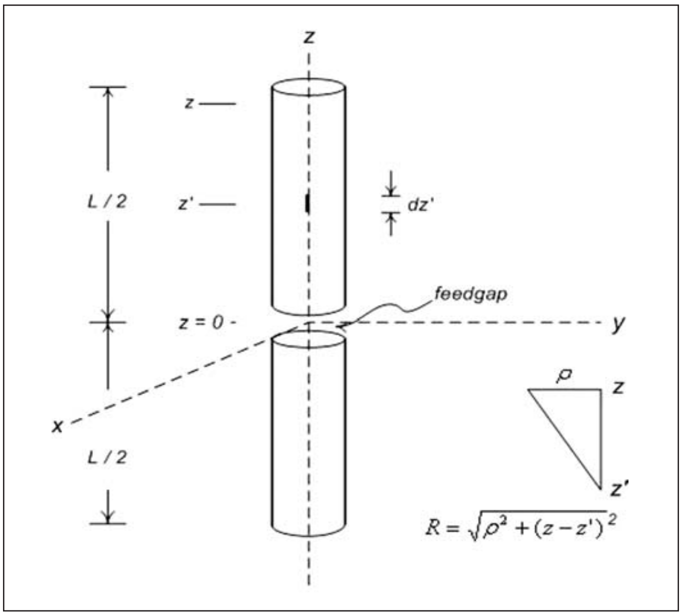

Figure 1 · Integral equation formulation.

The Method of Moments

The fundamental concept behind the MoM employs orthogonal expansions and linear algebra to reduce the integral equation problem to a system of simultaneous linear equations. This is accomplished by defining the unknown current distribution $I_z(z')$ in terms of an orthogonal set of "basis" functions and invoking the boundary conditions—the values of the electric field on the surface of the wire and in the feedgap—through the use of an inner product formulation. This inner product operation employs an orthogonal set of "testing" functions to enforce the boundary conditions, in an average sense, along the surface of the wire and in the feedgap. Moving the current's expansion coefficients to the outside of the integro-differential operator permits the evaluation of known functions, yielding values which are loosely defined as impedances. The current's expansion coefficients, the orthogonal projections of the electric field boundary conditions, and these so-called impedances are gathered into a system of simultaneous linear equations. This system of equations is solved to yield the current's expansion coefficients. The original current distribution is then determined by introducing these coefficents back into the basis function expansion.

The solution procedure begins by defining the unknown current distribution $I_z(z')$ in terms of an orthogonal set of basis functions. Two categories of basis functions exist. Sub-domain basis functions, significantly more popular in industry, subdivide the wire into small segments and model the current distribution on each segment by a simple geometrical construct, such as a rectangle, triangle, or sinusoidal arc. The amplitudes of these

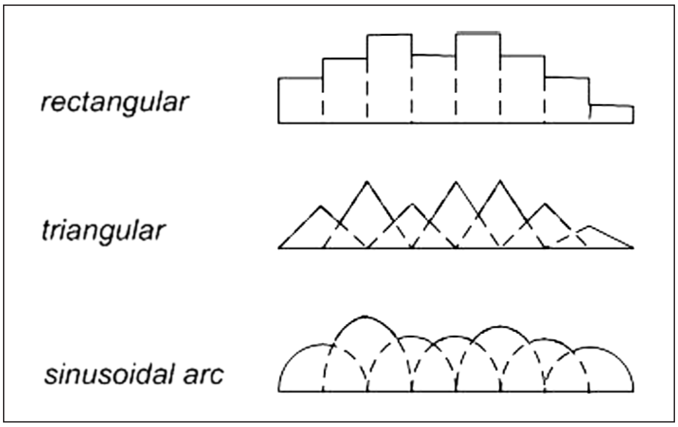

Figure 2 · Typical basis functions.

constructs represent the expansion function coefficients. These simple constructs, illustrated in Figure 2, often overlap to maintain continuity of the current distribution along the wire. Entire domain basis functions employ a more formal orthogonal expansion, such as a Fourier series, to represent the current distribution along the entire wire. Entire domain basis functions tend to yield more complicated calculations for the so-called impedances and, therefore, are less popular.

The introduction of the re-defined current distribution reduces the integral equation to the form

$$\sum_{n=1}^{N} C_n G_n(z) = E_z(z) \tag{5}$$

where

$$G_n(z) = \frac{1}{j4\pi\omega\varepsilon} \int_{-l/2}^{l/2} F_n(z') \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \frac{e^{-jkR}}{R} dz' \tag{6}$$

C_n = current's expansion coefficient

$F_n(z')$ = basis function

The boundary conditions are now enforced through the use of an inner product operator with a set of orthogonal testing functions. Each testing function is applied to both sides of the integral equation, the inner product then enforces the boundary condition at the location described by the testing function. This operation may be thought of as simply enforcing the boundary condition at a single point on the wire. After each testing function operation, the integral equation will appear as Equations 7 and 8.

$$\sum_{n=1}^{N} C_n < H_m(z), G_n(z) > = < H_m(z), E_z(z) > \tag{7}$$

where < > represents the inner product operator.

$$< H_m(z), G_n(z) > = \int_{-l/2}^{l/2} H_m(z) G_n(z) dz \tag{8}$$

where $H_m(z)$ is a testing function which has a non-zero value for only a small segment of wire located at $z = z_m$.

There are two common approaches to formulating the orthogonal set of testing functions. The first approach, the point matching or co-location technique, defines the testing function in terms of Dirac delta functions (Eq. 9).

$$H_m(z) = \delta(z - z_m) \tag{9}$$

where z_m are specific points on the wire at which the boundary conditions are enforced. The z_m are usually selected to correspond with the midpoint of each basis function. The second approach, Galerkin's technique, defines the testing function to be the same as the basis function. Galerkin's technique, although more complicated from a computational perspective, enforces the boundary condition more rigorously than the point matching technique. However, this more rigorous approach is seldom required for simple wire antenna problems.

The entire boundary condition is enforced by applying the complete set of testing functions. This operation yields a set of integral equations.

$$[Z_{mn}][I_n] = [V_m] \tag{10}$$

where

$$Z_{mn} = \int_{-l/2}^{l/2} H_m(z) G_n(z) dz \tag{11}$$

$$I_n = C_n \tag{12}$$

$$V_m = \int_{-l/2}^{l/2} H_m(z) E_z(z) dz \tag{13}$$

This circuit-like set of simultaneous linear equations will yield the value of C_n.

$$[I_n] = [Z_{mn}]^{-1} [V_m] \tag{14}$$

Limitations and Considerations

The validity of the assumptions introduced into MoM type formulations are established through empirical means. The codes incorporating these formulations are run for a large number of test cases with the results compared to experimental observation. Certain topics have received significant attention in the literature: the current distribution on the wire (the "thin wire approximation"), the orthogonality and completeness of the basis and testing functions, the modeling of the feedpoint excitation, the numerical evaluation of Z_{mn}, and the solution technique which yields C_n from the set of simultaneous linear equations. Although some of the assumptions continue to attract attention from a mathematically rigorous perspective, the codes incorporating them have been thoroughly exercised and deemed suitable for antenna engineering applications.

The most well-known of the codes using the MoM is the Numerical Electromagnetics Code (NEC), which is widely used to solve problems that can be defined as sets of one or more "wires" (linear elements).

Summary

The Method of Moments is a popular solution technique for integral equation problems found in engineering electromagnetics. This tutorial has attempted to present an outline of this technique from a practicing RF engineer's perspective with a minimum of mathematical rigor. The essential elements of integral equation formulation, basis and testing function definition, and reduction to a set of simultaneous linear equations, have been reviewed. The interested reader is referred to the many excellent textbooks on this subject, such as [2, 3, 4].

References

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